

# Collapsing shear-free perfect fluid spheres with heat flow

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## Abstract

A global view is given upon the study of collapsing shear-free perfect fluid spheres with heat flow. We apply a compact formalism, which simplifies the isotropy condition and the condition for conformal flatness. The formulas for the characteristics of the model are straight and tractable. This formalism also presents the simplest possible version of the main junction condition, demonstrated explicitly for conformally flat and geodesic solutions. It gives the right functions to disentangle this condition into well known differential equations like those of Abel, Riccati, Bernoulli and the linear one. It yields an alternative derivation of the general solution with functionally dependent metric components. We bring together the results for static and time-dependent models to describe six generating functions of the general solution to the isotropy equation. Their common features and relations between them are elucidated. A general formula for separable solutions is given, incorporating collapse to a black hole or to a naked singularity.

## 1 Introduction

Spherically symmetric radiating spacetimes are important in astrophysics for modelling radiating stars and in cosmology. Gravitational collapse is a highly

dissipative process, required to account for the enormous binding energy of the resulting object [1]. In the diffusion approximation this is described by the heat flux. It allows to join the interior solution to the Vaidya shining star exterior [2]. Radiating models are necessary in cosmology to describe the formation of structure, evolution of voids and the study of singularities [3].

Shear-free perfect fluids with heat flux are often studied in order to simplify the calculations and allow realistic analytic solutions. Two of their advantages are that there are just two metric components and their space evolution is governed by the isotropy condition, which is an ordinary second order linear differential equation in the radial variable [3],[4]. In the presence of heat flux the only non-trivial non-diagonal component of the Einstein equations becomes an expression for it. The vanishing of the heat flux implies a severe constraint, which transforms the isotropy condition into a non-linear and highly complicated differential equation with few explicit solutions [3],[5],[6],[7]. This situation persists even when anisotropy of the pressure is allowed both for shear-free and geodesic fluids [8],[9].

Many analytic solutions of the isotropy condition have been found. It has been written in a compact form already in 1948 [5]. Published in an obscure journal, this paper has been discussed nevertheless in some monographs [3],[7]. In spite of this, researchers prefer to work directly with the metric coefficients, which complicates the investigation. The isotropy condition is the same for static perfect fluid models and for time-dependent ones. In the static case both the heat flux and the off-diagonal component of the Einstein tensor vanish identically, so there is no additional constraint on the isotropy condition. Time dependence is obtained by promoting the integration constants into arbitrary functions of time. As a result, there are two groups of authors - the static (S) and the dynamical (D) group. Strangely enough, almost no interaction exists between them. This problem becomes especially annoying when the general solution of the isotropy equation is discussed. Recently, such a solution was proposed by D authors [10], using the Lie symmetries method. It was treated as a class of solutions at first, but later this statement was corrected [11]. However, generating functions have been found by the S group as early as 1971 [12],[13]. Interestingly, the S authors were not aware of previous research inside their group, so more such functions appeared during the years [14],[15],[16],[17],[18] (in the last reference a connection is made between it and the previous one). Of course, only non-vanishing in the static limit characteristics of the models were studied,

like energy density, pressure and the mass function.

While in the static case the main junction condition to the Schwarzschild solution is the vanishing of the pressure on the surface, in the dynamical case the correct joining was found in 1985 [2]. Like the isotropy condition, this is an ordinary differential equation. The difference is that it has only time derivatives and is non-linear. Once again, mainly its formulation in terms of the metric coefficients was considered [19],[20],[21]. During the solving process it has been found that some combinations simplify the computations.

One of the important questions of gravitational collapse is whether it ends in a black hole or a naked singularity. A model with separable metric was thoroughly studied [22],[23],[24] and it was found that a black hole forms at the end. However, there exists a simple solution of the same junction condition, when horizon never appears and the fate of the collapsing matter is a naked singularity [25]. Further, conformally flat and geodesic models were studied extensively, but the fact that the latter are a subclass of the first is not universally known.

In the present paper we address all these questions. In Sect.2 the field equations are given as well as the definitions of important characteristics of the fluid spheres like energy density, pressure, heat flux, expansion scalar, mass function, the second Weyl invariant and the distribution of the causal temperature. In Sect. 3 the junction conditions between the interior and the exterior are presented and the definitions of quantities that lie or depend on the surface are written. These are the surface luminosity, redshift and temperature, the luminosity at infinity and the total energy radiated during the collapse. It is stressed that the energy is stored in the integration functions. The condition for the formation of a horizon and consequently a black hole is also mentioned. Sect. 4 contains the compact formulation of the isotropy equation (LG formalism) and its general solution, obtained in six different ways. The relations between the different generating functions are elucidated and the characteristics of the model are expressed through some of them. In the next few sections several classes of solutions are derived utilizing the LG formalism. These are separable solutions in Sect. 5, conformally flat and geodesic solutions in Sect. 6 and solutions with functional dependence between the metric components in Sect. 7. For the sake of completeness we present chronologically the remaining solutions known to us in Sect. 8. The last section contains some conclusions.

## 2 Field equations

The collapse of a shear-free perfect fluid sphere is described by the following metric in isotropic comoving coordinates

$$ds^2 = -A^2 dt^2 + B^2 (dr^2 + r^2 d\Omega^2), \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad (1)$$

where  $A$  and  $B$  depend on  $r$  and  $t$ . The energy-momentum tensor for a fluid undergoing dissipation in the form of heat flow reads [4],[22]

$$T_{ik} = (\mu + p) u_i u_k + p g_{ik} + q_i u_k + q_k u_i, \quad (2)$$

where  $\mu$  is the energy density of the fluid,  $p$  is the isotropic pressure,  $u_i$  is the four-velocity and  $q_i$  is the radial heat flux vector, which is orthogonal to  $u_i$ . In comoving coordinates

$$u^i = A^{-1} \delta_0^i, \quad q^i = (0, q, 0, 0). \quad (3)$$

The non-trivial Einstein equations are

$$8\pi\mu = \frac{3\dot{B}^2}{A^2 B^2} - \frac{1}{B^2} \left( \frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB} \right), \quad (4)$$

$$8\pi p = \frac{1}{A^2} \left( -\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB} \right) + \frac{1}{B^2} \left( \frac{B'^2}{B^2} + \frac{2A'B'}{AB} + \frac{2A'}{rA} + \frac{2B'}{rB} \right), \quad (5)$$

$$8\pi p = \frac{1}{A^2} \left( -\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB} \right) + \frac{1}{B^2} \left( -\frac{B'^2}{B^2} + \frac{A'}{rA} + \frac{B'}{rB} + \frac{A''}{A} + \frac{B''}{B} \right), \quad (6)$$

$$8\pi q = \frac{2}{B^2} \left( \frac{\dot{B}}{AB} \right)'. \quad (7)$$

Here the dot and the prime stand for time and radial derivatives respectively. The rate of collapse  $\Theta$  is given by [19]

$$\Theta = \frac{3\dot{B}}{AB} \quad (8)$$

Let us use the variable  $u = r^2$  and organize the time derivative terms in terms of  $\Theta$ . The Einstein equations become

$$8\pi\mu = \frac{\Theta^2}{3} - \frac{4}{B^2} \left( \frac{3B_u}{B} - \frac{uB_u^2}{B^2} + \frac{2uB_{uu}}{B} \right), \quad (9)$$

$$8\pi p = -\frac{\Theta^2}{3} - \frac{2\dot{\Theta}}{3A} + \frac{4}{B^2} \left[ \frac{uB_u}{B} \left( \frac{B_u}{B} + \frac{2A_u}{A} \right) + \frac{A_u}{A} + \frac{B_u}{B} \right], \quad (10)$$

$$8\pi q = \frac{4r\Theta_u}{3B^2}, \quad (11)$$

$$\frac{2B_u^2}{B^2} + \frac{2A_uB_u}{AB} - \frac{A_{uu}}{A} - \frac{B_{uu}}{B} = 0. \quad (12)$$

The last equation is the difference between Eq (5) and Eq (6) and represents the isotropy of pressure. It contains no time derivatives and is an ordinary second-order differential equation for  $A, B$ . Thus we have the freedom to choose arbitrarily one of them and solve for the other. Time dependence arises when the integration constants  $C_i$  are promoted to integration functions  $C_i(t)$ . Eqs (8-11) become expressions for  $\Theta, \mu, p$  and  $q$ . The effective adiabatic index of the fluid  $\Gamma = d \ln p / d \ln \mu$  also can be found.

When the heat flow  $q$  vanishes Eq (11) becomes another condition on  $A, B$  and the combination with the isotropy condition leads to a non-linear equation with few solutions [3],[5], [6], [7]. This situation remains also in the anisotropic case for shear-free or geodesic collapsing spheres [8], [9].

The static case follows when the functions  $C_i(t)$  become constants again and  $\Theta = 0 = \dot{A}$ . Then Eq (11) yields simply  $q = 0$  but the isotropy condition remains the same. Thus to every dynamical model corresponds a static one.

An important characteristic in the general case is the mass function  $m$  of the fluid ball [19]

$$\frac{m}{r^3} = \frac{B}{2} \left( \frac{\dot{B}^2}{A^2} - \frac{2B'}{rB} - \frac{B'^2}{B^2} \right) = \frac{B^3\Theta^2}{18} - 2B \left( \frac{B_u}{B} + \frac{uB_u^2}{B^2} \right). \quad (13)$$

The conformal tensor has one essential component, which is given in an invariant way by the second Weyl invariant  $\Psi_2$ . It can be determined from the following formula, holding for anisotropic fluid spheres with heat flow [8]

$$\frac{m}{R^3} = \frac{4\pi}{3} (\mu + p_t - p_r) - \Psi_2. \quad (14)$$

In the isotropic and shear-free case  $R = rB$  and we have

$$\Psi_2 = \frac{8\pi\mu}{6} - \frac{m}{r^3B^3}. \quad (15)$$

Eqs (9,13) yield

$$\Psi_2 = \frac{4u}{3B^2} \left[ \left( \frac{B_u}{B} \right)^2 - \left( \frac{B_u}{B} \right)_u \right]. \quad (16)$$

The terms with time derivatives cancel and the equation for conformally flat solutions  $\Psi_2 = 0$  is an ordinary differential equation in  $u$ , like the isotropy condition (12). Formula (14) has been derived originally in terms of the electric part of the Weyl tensor  $E$  [26]. Comparing both formulas one comes to the conclusion that  $E = 3\Psi_2$ .

Another characteristic of the collapsing sphere is the temperature distribution  $T$  among its volume. Unlike the previous quantities, it is determined by a non-linear differential causal transport equation. For the present metric it becomes [20]

$$\tau (qB)^\cdot + qAB = -\frac{\kappa (AT)'}{B}, \quad (17)$$

where  $\kappa$  is the thermal conductivity and  $\tau$  is the relaxational time-scale which gives rise to the causal behaviour of the theory. Both of them depend on the temperature in general. A physically reasonable choice is the transportation of energy by massless particles. Then Eq (17) becomes

$$\frac{\alpha}{4}X' + qA^{4-\sigma}B^2X^{\sigma/4} + \beta (qB)^\cdot A^3B = 0, \quad X = (AT)^4, \quad (18)$$

where  $\alpha, \beta, \sigma$  are constants. In the non-causal case  $\beta = 0$  it is easily solved. Integrable causal cases are  $\sigma = 0; 2; 4$  [1],[20],[27],[28],[29]. Explicit solutions do not alter the structure of this equation.

### 3 Junction conditions

The collapsing fluid lies within the sphere  $\Sigma$  defined by  $r = r_\Sigma$ . The fluid is radiating, hence, the exterior is not vacuum, but the outgoing Vaidya spacetime with the metric [4]

$$ds^2 = -\left(1 - \frac{2M(v)}{\rho}\right)dv^2 - 2dv d\rho + \rho^2 d\Omega^2. \quad (19)$$

The junction conditions represent the continuity of the first and second fundamental forms on  $\Sigma$ . This results in the relations [2]

$$p = qB, \quad M(v) = m(r, t), \quad \rho(v) = rB \quad (20)$$

which hold on the surface  $\Sigma$ . Here  $M$  is the total mass of the radiating sphere,  $\rho_\Sigma$  is its radius as seen from outside and  $v$  is the time of the distant

observer. The first relation is a differential equation containing derivatives with respect to  $t$  only. Replacing Eqs (10,11) and setting  $r = r_\Sigma$  we get

$$\Theta^2 + \frac{2\dot{\Theta}}{A} + \frac{4r\Theta_u}{B} = 24\pi p_s, \quad (21)$$

where  $p_s$  is the pressure of the corresponding static model.

Some of the characteristics of the model are defined on its surface. Such are the surface luminosity  $\Lambda_\Sigma$  and the redshift  $z_\Sigma$  [4],[30]

$$\Lambda_\Sigma = \frac{2}{3}u_\Sigma^{3/2}(B\Theta_u)_\Sigma, \quad z_\Sigma = \left(1 + \frac{2uB_u}{B} + \frac{rB\Theta}{3}\right)_\Sigma^{-1} - 1. \quad (22)$$

The exterior time  $v$  is related to the interior one  $t$  as follows

$$v = \int (1 + z_\Sigma) A_\Sigma dt. \quad (23)$$

The surface temperature of the star is

$$T_\Sigma^4 = \frac{\Lambda_\Sigma}{4\pi\delta\rho_\Sigma^2} = \left(\frac{r\Theta_u}{6\pi\delta B}\right)_\Sigma, \quad (24)$$

where  $\delta$  is some constant. The total luminosity for an observer at rest at infinity reads

$$\Lambda_\infty = -\frac{dM}{dv} = \frac{\Lambda_\Sigma}{(1 + z_\Sigma)^2} = \frac{2}{3}u_\Sigma^{3/2}B\Theta_u \left(1 + \frac{2uB_u}{B} + \frac{rB\Theta}{3}\right)_\Sigma^2, \quad (25)$$

The total energy radiated during the collapse of the fluid sphere follows from the previous equation

$$E_\infty = \int_{v_b}^{v_e} \Lambda_\infty dv = M(v_b) - M(v_e), \quad (26)$$

where  $v_b$  ( $v_e$ ) is the exterior time of the collapse's start (end). These correspond to  $t_b$  ( $t_e$ ) according to Eq (23). Eqs (13,20) show that the radiated energy results from the change in the integration functions  $\Delta C_i = C_i(t_b) - C_i(t_e)$ . Thus the energy that a static model can give out is stored in its constants, which are animated during the collapse and evolve under the single condition given by Eq (21).

A separable solution was discussed in detail [22],[23],[24]. There  $t_b = -\infty$  and the exterior solution at that moment is the static exterior Schwarzschild solution in isotropic coordinates

$$ds^2 = - \left( \frac{1 - M_0/2\rho}{1 + M_0/2\rho} \right)^2 dt^2 + \left( 1 + \frac{M_0}{2\rho} \right)^4 (d\rho^2 + \rho^2 d\Omega^2). \quad (27)$$

Here  $M_0$  is the constant total mass. The sphere collapses until a black hole is formed at  $t_e = t_{BH}$ . This happens when the coefficient  $g_{00}$  in the exterior metric (19) vanishes at the surface  $\Sigma$  and a horizon appears. Eq (20) shows that the relation  $2m = rB$  should be satisfied there. It may be written with the help of Eq (13) as follows

$$\left( 1 + \frac{2uB_u}{B} + \frac{rB\Theta}{3} \right)_\Sigma \left( 1 + \frac{2uB_u}{B} - \frac{rB\Theta}{3} \right)_\Sigma = 0. \quad (28)$$

The first multiplier vanishes in order to satisfy this relation. Then Eqs (22, 25) lead to the blowing up of the redshift and vanishing of the luminosity at infinity.

Another separable solution with  $t_b = -\infty$ , however, never develops a horizon [25] and the collapse proceeds till all the mass is burnt out, namely  $M(t_e) = 0$ , which maximizes  $E_\infty$ . The final state can also be any static model for anisotropic fluids with vanishing radial pressure  $p_r = 0$  and no heat flow [31]. In this case Eq (21) is trivially satisfied.

## 4 Six generating functions

All of the previous formulas are based on the solution of Eq (12). The search for its solutions spans an interval of 63 years, starting from 1948 when Kustaanheimo and Qvist wrote it in a very compact form [3],[5],[7]. Introducing instead of  $A$  and  $B$  the potentials  $L = 1/B$  and  $G = A/B$  it becomes

$$2GL_{uu} = LG_{uu}, \quad B = 1/L, \quad A = G/L. \quad (29)$$

This is a linear second-order differential equation. One can choose an ansatz for  $G$  and solve for  $L$  or vice versa. However, a general solution is hard to find. Choosing the function  $K = L_u/L$  transforms Eq (29) into a Riccati equation for  $K$

$$K_u + K^2 - \frac{G_{uu}}{2G} = 0, \quad (30)$$



which is first order, but still a general solution for any  $G$  is not known.

This difficulty may be overcome if we choose one of the potentials as a function of  $AB$  or its  $u$ -derivative. For example let us take  $A$  and  $W = 1/AB$ . Then

$$L = AW, \quad G = A^2W \quad (31)$$

and Eq (29) becomes

$$\frac{A_u}{A} = \pm \sqrt{\frac{W_{uu}}{2W}}, \quad B = 1/AW. \quad (32)$$

It is readily integrable when an arbitrary  $W$  is given [10].

$$A = A_0(t) \exp \pm \int \sqrt{\frac{W_{uu}}{2W}} du. \quad (33)$$

The function of integration  $A_0(t)$  may be removed from  $A$  by a time change but it remains in the expression for  $B$ . In the above reference the solution was presented as a result of Lie symmetries analyses, together with other classes of solutions. Later it was emphasized that this is a general solution [11] and in this way the potential of Msomi, Govinder and Maharaj  $W$  is a generating function for the metric  $A, B$  and all of the above characteristics of the model can be expressed through it. In addition,  $W$  together with Eqs (32-33) comprises the general solution of the isotropy condition Eq (29). The potential  $\hat{W} = W^{-1} = AB$  was also discussed [10] with similar expressions for  $A, B$ .

The story does not begin here, however. As emphasized, the isotropy condition holds also for the static case, which was studied extensively in the past. Unfortunately, most of the authors worked in the so-called curvature coordinates. Yet there is some amount of papers in isotropic coordinates. Quite a few concrete solutions have been found and beside them five other generating functions. As we pointed out in the introduction, the work of this 'static' group of authors is completely unknown to the 'dynamical' group, which studied the time-dependent metric. The opposite is also true. One of the purposes of the present paper is to present both the static and dynamical results together, so that the future efforts may be united and rediscoveries avoided. We shall derive the generating functions from one another. Amazingly, their authors were unaware of the work of each other with one small exception.

Thus, let us take instead of  $W$  the potential

$$U = -\frac{2W_u}{W} = 2(\ln AB)_u, \quad (34)$$

$$U_u = \frac{1}{2}U^2 - H^2, \quad H^2 = \frac{4A_u^2}{A^2}. \quad (35)$$

In this form the isotropy condition was given by Kuchowicz [12],[13],[32]. For  $U$  this is a Riccati equation, but for  $H$  is an algebraic one and for  $A$  is a simple linear equation. The metric is given by

$$A = \exp \frac{1}{2} \int H du, \quad B = B_0(t) \exp \frac{1}{2} \int (U - H) du, \quad (36)$$

where  $B_0$  is a function of integration. Thus  $U$  is another generating function. The change  $H^2 = UJ$  turns the Riccati equation for  $U$  into a Bernoulli equation, which is integrable, or into a simple algebraic equation for  $J$

$$U_u = \frac{1}{2}U^2 - JU. \quad (37)$$

Here one can choose either  $U$  or  $J$  as a generating function.

Let us replace next the potentials  $U, H$  by  $f, g$  according to the following expressions

$$U = \frac{1}{f}, \quad H = \frac{g}{f}, \quad (38)$$

so that

$$\frac{1}{f} = 2(\ln AB)_u, \quad g = \frac{(\ln A)_u}{(\ln AB)_u}. \quad (39)$$

Then the isotropy condition becomes simply

$$f_u = g^2 - \frac{1}{2} \quad (40)$$

and one can choose either  $f$  or  $g$  as a generating function. The other one follows immediately, while the metric is given by

$$A = \exp \frac{1}{2} \int \frac{g}{f} du, \quad B = B_0 \exp \frac{1}{2} \int \frac{1-g}{f} du. \quad (41)$$

This third generating function was found by Goldman [14], who also gave some particular solutions. His work was corrected and further developed by

Knutsen [16]. The latter author expressed the characteristics of the static model in term of the potentials, studied the energy conditions and proposed another particular solution.

Next, let us take the potential

$$\Phi = \ln AB = \int \frac{du}{2f}. \quad (42)$$

Then Eq (40) yields

$$g = \varepsilon \frac{\sqrt{\Phi_u^2 - \Phi_{uu}}}{\sqrt{2}\Phi_u}, \quad (43)$$

where  $\varepsilon = \pm 1$  and the metric is expressed through  $\Phi$

$$A = \exp \varepsilon \frac{\sqrt{2}}{2} \int \sqrt{\Phi_u^2 - \Phi_{uu}} du, \quad B = B_0 \exp \left( \Phi - \varepsilon \frac{\sqrt{2}}{2} \int \sqrt{\Phi_u^2 - \Phi_{uu}} du \right). \quad (44)$$

These are essentially the expressions of Ref.[18] and  $\Phi$  is the fourth generating potential, proposed by Lake.

The fifth one  $Z$  was introduced by Rahman and Visser [17]. It is obtained from the relation

$$\Phi = 2 \int \frac{Z}{1 - uZ} du \quad (45)$$

and is connected to the Goldman-Knutsen potentials by the equations

$$2f = \frac{1}{Z} - u, \quad g^2 = -\frac{Z_u}{2Z^2}. \quad (46)$$

Inserting them in Eq (41) we get for the metric

$$A = \exp \pm \frac{1}{\sqrt{2}} \int \frac{\sqrt{-Z_u}}{1 - uZ} du, \quad B = B_0 A^{-1} \exp \int \frac{Z}{1 - uZ} du. \quad (47)$$

Finally, we should mention the potentials introduced by Stewart [15]

$$P = 2r (\ln AB)_u, \quad S = 2r \left( \ln \frac{A}{B} \right)_u, \quad (48)$$

which satisfy the equation

$$2rP_u - \frac{P}{r} - \frac{1}{2}P^2 + SP + \frac{1}{2}S^2 = 0. \quad (49)$$

It can be shown that

$$P = \frac{r}{f}, \quad S = (2g - 1)P \quad (50)$$

and then Eq (49) transforms into Eq (40). In this way  $P$  is the sixth (and the last known to us) generating function.

Eq (40) shows that to every function  $f$  correspond two functions  $\pm g$ , Suppose we have a solution  $A_1, B_1$  with potentials  $f_1, g_1$ . Then there is another solution with  $f_2 = f_1$  and  $g_2 = -g_1$ . Eq (41) yields

$$A_2 = A_1^{-1}, \quad B_2 = B_1 A_1^2. \quad (51)$$

This is exactly the Buchdahl theorem [33] in the spherically symmetric case.

Now let us give the characteristics of the fluid model in terms of  $L$  and  $G$ :

$$ds^2 = L^{-2} \left( -G^2 dt^2 + dr^2 + r^2 d\Omega^2 \right), \quad (52)$$

$$\Theta = -\frac{3\dot{L}}{G}, \quad 8\pi q = \frac{4}{3}rL^2\Theta_u, \quad (53)$$

$$8\pi\mu = \frac{\Theta^2}{3} + 12L_u(L - uL_u) + 8uLL_{uu}, \quad (54)$$

$$8\pi p = -\frac{\Theta^2}{3} - \frac{2\dot{\Theta}L}{3G} + \left( \frac{4G_uL}{G} - 6L_u \right) (L - 2uL_u) - 2LL_u, \quad (55)$$

$$m = \frac{r^3}{L^3} \left[ \frac{\Theta^2}{18} + 2L_u(L - uL_u) \right], \quad \Psi_2 = \frac{4}{3}uLL_{uu}, \quad (56)$$

$$\Lambda_\Sigma = \frac{2}{3} \left( \frac{r^3\Theta_u}{L} \right)_\Sigma, \quad T_\Sigma^4 = \frac{1}{6\pi\delta} (r\Theta_u L)_\Sigma, \quad (57)$$

$$z_\Sigma = \left( \frac{2uL_u + r\Theta/3}{L - 2uL_u - r\Theta/3} \right)_\Sigma, \quad (58)$$

$$\Lambda_\infty = \frac{\Lambda_\Sigma}{L_\Sigma^2} (L - 2uL_u - r\Theta/3)_\Sigma^2, \quad (59)$$

It is clear that  $G$  appears only through  $\Theta$ , except in the pressure. Note also the simple formula for the second Weyl invariant.

Finally, let us express the characteristics in terms of a generating function. The most convenient are the Goldman-Knutsen potentials.  $A$  and  $B$  are found from Eq (41). The rest are

$$\Theta = \frac{3}{2A} \int \left( \frac{1-g}{f} \right) du, \quad (60)$$

$$8\pi\mu = \frac{\Theta^2}{3} - \frac{1}{f^2 B^2} \left[ (1-g) (6f + 3u - ug - 4ug^2) - 4ufg_u \right], \quad (61)$$

$$8\pi p = -\frac{\Theta^2}{3} - \frac{2\dot{\Theta}}{3A} + \frac{1}{f^2 B^2} \left[ 2f + u (1 - g^2) \right], \quad (62)$$

$$\frac{m}{(rB)^3} = \frac{\Theta^2}{18} + \frac{g-1}{2f^2 B^2} [2f + u(1-g)], \quad (63)$$

$$\Psi_2 = \frac{u}{3f^2 B^2} \left[ (1-g) (2 + g - 2g^2) + 2fg_u \right], \quad (64)$$

$$\Lambda_\infty = \frac{\Lambda_\Sigma}{(1+z_\Sigma)^2}, \quad z_\Sigma = \left( 1 + \frac{1-g}{f} u + rB\Theta/3 \right)_\Sigma^{-1} - 1, \quad (65)$$

while  $q$ ,  $\Lambda_\Sigma$  and  $T_\Sigma$  are given by Eqs (11,22,24) respectively. In the junction condition (21) one should put

$$8\pi p_s = \frac{1}{f^2 B^2} \left[ 2f + u (1 - g^2) \right], \quad (66)$$

computed at the surface  $\Sigma$ . Here  $g$  is given by Eq (41) and  $f(r, t)$  is an arbitrary function of  $u$  and a number of functions  $C_i(t)$ . When the latter and  $B_0(t)$  are constant,  $q, \Theta, \Lambda_\Sigma, \Lambda_\infty, T_\Sigma, T, p_\Sigma$  vanish, so that the model becomes static. Then the formulas for the energy density, the pressure and the mass coincide with those of Knutsen [16].

## 5 Separable solutions

Separable solutions have the following metric [34]

$$A = j(t) \alpha(u), \quad B = h(t) \beta(u). \quad (67)$$

The function  $j(t)$  may be set to 1 by a time change. Then Eq (12) shows that  $\alpha$  and  $\beta$  should satisfy the isotropy condition as a static metric. Thus the generating functions are static, while  $h(t) = B_0(t)$ . Replacing the above metric in the expressions for the various characteristics we obtain

$$\Theta = \frac{3\dot{h}}{\alpha h}, \quad 8\pi q = -\frac{4r\alpha_u \dot{h}}{\alpha^2 \beta^2 h^3}, \quad (68)$$

$$\Lambda_\Sigma = -2\dot{h} \left( \frac{r^3 \alpha_u \beta}{\alpha^2} \right)_\Sigma, \quad T_\Sigma^4 = -\frac{\dot{h}}{2\pi\delta h^2} \left( \frac{r\alpha_u}{\alpha^2\beta} \right)_\Sigma, \quad (69)$$

$$8\pi\mu = \frac{3\dot{h}^2}{\alpha^2 h^2} + \frac{8\pi\mu_s}{h^2}, \quad 8\pi p = -\frac{\dot{h}^2 + 2h\ddot{h}}{\alpha^2 h^2} + \frac{8\pi p_s}{h^2}, \quad (70)$$

$$m = \frac{(r\beta)^3}{2\alpha^2} h\dot{h}^2 + hm_s, \quad \Psi_2 = \frac{\Psi_{2s}}{h^2}, \quad (71)$$

$$\Lambda_\infty = \Lambda_\Sigma \left( 1 + \frac{2u\beta_u}{\beta} - \frac{r\beta\dot{h}}{\alpha} \right)_\Sigma^2, \quad (72)$$

$$z_\Sigma = \left( \frac{-2u\alpha\beta_u + r\beta^2\dot{h}}{\alpha\beta + 2u\alpha\beta_u - r\beta^2\dot{h}} \right)_\Sigma, \quad (73)$$

where quantities with an index  $s$  correspond to the static model with metric  $\alpha, \beta$ .

We suppose that the boundary of the static model is also at  $r_\Sigma$ , so there  $p_s = 0$  and Eq (21) becomes

$$2h\ddot{h} + \dot{h}^2 - 2a\dot{h} = 0, \quad a = 2 \left( \frac{r\alpha_u}{\beta} \right)_\Sigma. \quad (74)$$

This ordinary second-order differential equation governs the behaviour of  $h$ . Its first integral reads

$$\dot{h} = \frac{2}{\sqrt{h}} (a\sqrt{h} - b), \quad (75)$$

where  $b$  is an integration constant. Integrating once more gives

$$t - t_0 = \frac{h}{2a} + \frac{b}{a^2} \sqrt{h} + \frac{b^2}{a^3} \ln \left| \sqrt{h} - \frac{b}{a} \right|. \quad (76)$$

Here  $t_0$  is a second integration constant. When  $b \neq 0$  one can enforce the equality  $b = a$  by absorbing a constant in  $\beta$  [22],[23],[24],[28],[29]. This solution was analysed in great detail and leads to the formation of a black hole.

When  $b = 0$  the solution is  $h = 2a(t - t_0)$  [25]. In this reference this particular solution was given as a simple ansatz satisfying Eq (74), the general solution being unavailable, according to the authors. Here we have derived it

from the general formalism. It is easy to see that  $m$  and  $rB$  are proportional to  $t - t_0$ , hence, their ratio is constant. At  $\Sigma$

$$\left(\frac{2m}{rB}\right)_{\Sigma} = 2 \left( \frac{8u^2\alpha_u^2}{\alpha^2} + \frac{m_s}{r\beta} \right)_{\Sigma}. \quad (77)$$

This certainly can be made less than unity by choosing the constants of the arbitrary static solution. Going back to the arguments that lead to Eq (28), one comes to the conclusion that no horizon is formed during the process of collapse. Its end is marked by  $t_e = t_0$  when the mass burns out completely and vanishes. The energy accumulated during the collapse is radiated at the same rate. Both luminosities, the surface temperature and redshift are constant, while  $\mu, p$  and  $\Psi_2$  diverge as  $(t - t_0)^{-2}$ ,  $q \sim (t - t_0)^{-3}$  and  $\Theta \sim (t - t_0)^{-1}$ . This may be an indication for the formation of a naked singularity. In Ref. [25] the special solution

$$\alpha = 1 + cu, \quad \beta = 1 \quad (78)$$

was considered, which in addition is conformally flat. Solutions with no horizon appear also in higher dimensional spacetimes [35],[36].

## 6 Conformally flat solutions

These solutions have  $\Psi_2 = 0$  and a look at Eq (56) shows that the LG formalism is the most appropriate for their study. Eqs (29,56) yield

$$L_{uu} = 0, \quad G_{uu} = 0. \quad (79)$$

Integration produces four integration functions. In the general case they are independent and Eq (52) indicates that in  $G$  one of them may be set to unity. Hence

$$L = C_1(t)u + C_2(t), \quad G = u + C_3(t). \quad (80)$$

The metric and the combination  $AB$ , which is the basis of the six generating functions, become

$$A = \frac{u + C_3}{C_1u + C_2}, \quad B = \frac{1}{C_1u + C_2}, \quad AB = \frac{u + C_3}{(C_1u + C_2)^2}. \quad (81)$$

The characteristics of the fluid model depend on the three functions  $C_i(t)$ . It is much simpler to use  $C_1, L$  and  $G$  instead. One should note that

$$L_u = C_1, \quad G_u = 1, \quad \Theta_u = -\frac{3\dot{C}_1}{G} + \frac{3\dot{L}}{G^2}. \quad (82)$$

We can use Eqs (52-59) for the characteristics of the model, inserting in them the above formulas. Eq (56) yields

$$m = \frac{4\pi\mu r u}{3L^3}, \quad (83)$$

which follows also from Eq (15) or the square of the conformal tensor [19]. The time evolution of  $C_1, L$  and  $G$  is governed by the junction condition Eq (21) on  $\Sigma$ , where  $p_s$  is given by Eq (55) with  $\Theta = 0$ . Dropping for a while the index  $\Sigma$ , we obtain after a lengthy calculation

$$\begin{aligned} 2L\dot{L}\dot{G} + (3\dot{L}^2 - 2L\ddot{L})G + 4rL\dot{L}G - 4r\dot{C}_1LG^2 - 4(L - 2uC_1)LG^2 + \\ + (8C_1L - 12uC_1^2)G^3 = 0. \end{aligned} \quad (84)$$

With respect to  $G$  this is an Abel equation of the first kind

$$A_1\dot{G} + A_2G + A_3G^2 + A_4G^3 = 0. \quad (85)$$

It is not soluble analytically in general, but in some degenerate cases one obtains [21] an algebraic equation (when  $A_1 = 0$ ) or Bernoulli equations (when  $A_3$  or  $A_4$  vanishes), which are integrable.

With respect to  $C_1$  this is a Riccati equation

$$R_1\dot{C}_1 + R_2C_1 + R_3C_1^2 + R_4 = 0, \quad (86)$$

which may be transformed into a inhomogeneous second-order linear equation. The general solution may be found if a particular one is known.

Finally, in order to elucidate the character of Eq (84) with respect to  $L$  we make the replacement

$$L = l^{-2}, \quad (87)$$

which gives

$$G\ddot{l} - (\dot{G} + 2rG)\dot{l} - G^2l - rG^2l^3\dot{C}_1 + 2G^2(u + G)l^3C_1 -$$



$$-3uG^3l^5C_1^2 = 0. \quad (88)$$

One can make this equation linear and homogenous in  $l$  in two ways. First, we simply put  $C_1 = 0$  [19], [20]. When  $G$  is constant there are three classes of solutions. In these references  $G$  was taken in the form  $G = C_3u + 1$ , which leads to more involved coefficients of the equations. They hold for  $u$  in general, but are needed for  $u_\Sigma$  only. Another possibility is to make  $L \sim C_1$ . This happens when  $C_2 = kC_1$  with  $k$  some coefficient. The case of constant  $G$  was studied [1]. In fact, in these two cases  $A$  and  $B$  become separable and Eq (88) is the analogue of the integrable Eq (74). Some other conformally flat and separable solutions with a single, linear in time, integration function have been discussed [25],[37].

A well known example of a static conformally flat solution is the interior Schwarzschild metric. In isotropic coordinates it looks like [17], [38]

$$A = \frac{1 + \frac{1+c_2}{c_2} \frac{u}{c_1}}{1 + u/c_1}, \quad B = \frac{1}{1 + u/c_1} \quad (89)$$

and has two constants. The limit  $c_2 \rightarrow \infty$  leads to the Einstein universe, while  $c_2 = -1/2$  yields the De Sitter universe [17]. The  $Z$  potentials of these solutions were given in the last reference. As pointed out in the beginning, one can 'animate' these classical static models by making the constants time-dependent. A model where they, in addition, depend on each other was given by Kramer [39] and studied later [40]. The junction condition becomes a complicated nonlinear second-order differential equation, which surprisingly may be solved in terms of a special function. Due to their simple structure, conformally flat solutions were among the first to be discovered [3],[41],[42],[43],[44],[45],[46]. The subclass with  $G = 1$  was studied too [47].

An important class of solutions are the geodesic or non-accelerating ones, which have  $A = 1$ . This leads to  $G = L$  and the metric becomes

$$ds^2 = -dt^2 + L^{-2} (dr^2 + r^2 d\Omega^2). \quad (90)$$

Eq (29) gives  $L_{uu} = 0$  and Eq (56) shows that the geodesic solutions are a subclass of the conformally flat solutions. We have for the expansion

$$\Theta = -\frac{3\dot{L}}{L}, \quad \Theta_u = -3 \left( \frac{C_1}{L} \right). \quad (91)$$

The characteristics of the model are obtained when these formulas are inserted in Eqs (53-59). For example, the pressure reads

$$8\pi p = -\frac{\Theta^2}{3} - \frac{2\dot{\Theta}}{3} - 4LC_1 + 4uC_1^2. \quad (92)$$

The condition  $A = 1$  simplifies the generating functions of geodesic models. Thus  $W = L$ ,  $H = 0$ ,  $g = 0$ ,  $\Phi = -\ln L$ ,  $Z = \text{const}$  and  $P = -S$ .

The junction condition Eq (21) becomes

$$-4rL^2\dot{C}_1 + 4L(r\dot{L} + L^2)C_1 - 4uL^2C_1^2 = 2L\ddot{L} - 5\dot{L}^2, \quad (93)$$

taken on the surface  $\Sigma$ . It differs substantially from Eq (84), although it still represents a Riccati equation for  $C_1$  [48]. When some of the coefficients are made to vanish, a Bernoulli or a soluble Riccati equation follows. The ansatz

$$L = L_0(t + t_0)^n, \quad (94)$$

where  $L_0, t_0, n$  are constants, transforms Eq (93) into the linear second-order equation of the confluent hypergeometric function. Elementary solutions were found for  $n = 0; -2/3; -2$  [48]. This method also reproduces the solution with

$$B = \frac{c_1}{2c_2} \left( \frac{1 - c_2c_3 \exp s}{1 - uc_3 \exp s} \right) s^2, \quad s = \left( \frac{6t}{c_1} \right)^{1/3}, \quad (95)$$

where  $c_i$  are constants. It has been discussed extensively in the past [27],[29], [49],[50],[51]. Finally, it should be mentioned that the study of collapsing shear-free perfect fluid models with heat flow began with the well-known Robertson-Walker cosmological model by promoting its constant  $k$  to a function  $k(t)$  [3],[41], [45],[52],[53].

## 7 Functional dependence between $A$ and $B$

A class of solutions to the isotropy condition has the functional dependence  $A(B)$ . The latter is equivalent to  $G(L)$ . Then Eq (29) becomes

$$2GL_{uu} = G_{LL}LL_u^2 + G_L LL_{uu}, \quad (96)$$

which may be written as

$$\frac{G_{LL}}{\frac{2G}{L} - G_L} L_u = \frac{L_{uu}}{L_u}. \quad (97)$$

Integrating once we obtain

$$L_u = C_1(t) \exp \int \frac{G_{LL}}{\frac{2G}{L} - G_L} dL, \quad (98)$$

where  $C_1(t)$  is an integration function. A second integration gives

$$\int \exp \left( - \int \frac{G_{LL}}{\frac{2G}{L} - G_L} dL \right) dL = C_1(t) u + C_2(t), \quad (99)$$

$C_2(t)$  being a second integration function. Choosing an explicit  $G(L)$  one obtains after the integrations an explicit or implicit expression for  $L(t, u)$ . The result for the variables  $A, L$  is similar

$$\int \exp \left( - \int \frac{2A_L + LA_{LL}}{A - LA_L} dL \right) dL = C_1(t) u + C_2(t). \quad (100)$$

This formula was found long ago [43] and considered to be the general solution of the isotropy equation. Formally this is true, because time plays the role of a parameter and not a variable in it. Therefore, effectively,  $A = A(u)$  and  $B = B(u)$ . Inverting the second equality and replacing it in the first, one finds that for every solution indirectly  $A = A(L)$ . Thus  $L$  may be considered as another generating function. However, the inversion and the double integration in Eqs (99,100) make the procedure rather cumbersome and implicit. Formula (100) was later rediscovered by the Lie symmetry method [10] and a few examples were presented for illustration in both references. In the LG formalism one of them is given by  $G = L^3$ . A brief calculation yields from Eq (99)

$$L = [C_1(t) u + C_2(t)]^{1/7}. \quad (101)$$

When  $A$  is a function of  $B$  (or  $L$ ) such is the base for the generating functions  $AB$ . Then Eq (39) shows that the Goldman-Knutsen potential  $g = g(L)$ . However,  $f$  depends on  $L$  and  $L_u$  in the general case.

A model of the same type was found in an attempt to generalize the exterior Schwarzschild solution (27) [54],[55]. It has

$$A = \frac{1 - F}{1 + F}, \quad B = B_0(t) (1 + F)^2, \quad F = \frac{c_1}{(1 + c_2 u)^{1/2}}. \quad (102)$$

Bayin [56] studied static solutions in isotropic coordinates of the type

$$A = A_0 \phi^{-c_1}, \quad B = B_0 \phi^{c_2} \quad (103)$$

with

$$\phi = c_3 e^{c_4 u}, \quad \phi^{1-c} = c_3 u + c_4, \quad (104)$$

where  $c = c(c_1, c_2)$  in a specified way. They also have  $A$  and  $B$  directly dependent on each other.

## 8 Other solutions

As stated before, static and time-dependent solutions are on equal footing with respect to the isotropy condition. We present only one of the metric functions in most cases, preserving the original notation. The constants below are understood. Chronologically, the first solutions were given by Narlikar et al in 1943 [7],[13],[57]. They are static and have

$$L_I = C_1 r^{1+n/2} + C_2 r^{1-n/2}, \quad L_{II} = C_3 r^{-k/2}, \quad (105)$$

where  $n > 0, 0 \geq k \geq -2$ . There are three cases of  $A$  for each  $L$ , including the interior Schwarzschild metric (SIM). Nariai [7],[58] found five static solutions, one of them coinciding with  $L_I$  from above. The rest are

$$L_I^2 = (a + bu)^\alpha, \quad L_{II}^2 = u(a + b \ln r)^2, \quad L_{III}^2 = a \cos(b + cu), \\ A_{IV}^2 = (a + bu)^\alpha. \quad (106)$$

In 1968 Strobel [3],[41] gave a list of cases of  $L_{uu}/L$  for which a solution of Eq (29) can be found in the handbooks on differential equations and two explicit solutions with  $L_{uu} = 0$ . This reference remained unnoticed. Kuchowicz [7],[12],[13],[32],[59] gave a host of new static solutions, using his generating function. In view of their great number we direct the reader to the original references. Goldman [14] studied three explicit static examples of his potential  $g$

$$g_I = \frac{a}{1 - bu}, \quad g_{II} = \frac{1}{\sqrt{2}} \coth(a - bu), \quad g_{III} = \cosh(a + bu). \quad (107)$$

Stewart [7],[15] applied the Buchdahl theorem to SIM to obtain a new static solution with

$$A^2 = c(1 + au)^2(1 - bu)^{-2}. \quad (108)$$

Sanyal and Ray [43] gave their Case 1 dynamical solution

$$A = C(t)u + D(t) \quad (109)$$

as a complementary one to their general solution (100). Modak [44] proposed a time-dependent metric, which coincides with the animated fourth Nariai solution. Pant and Sah [60] studied in detail the static model with

$$A = A_0 \frac{1 - k\delta}{1 + k\delta}, \quad B = \frac{(1 + k\delta)^2}{1 + u/a^2}, \quad \delta(u) = \frac{(1 + u/a^2)^{1/2}}{(1 + bu/a^2)^{1/2}}, \quad (110)$$

which is a generalisation of the Buchdahl solution (102) for  $b \neq 0$ .

Deng in 1989 invented a powerful method (Deng's ladder) [3],[45] for generating an infinite chain of more and more complicated solutions, varying with time, from a simple seed  $A_1$ . One finds next the general form of  $L_1$ , takes it as a seed, finds the general  $A_2$  and so on. He delivered the most general conformally flat solution and some others. Banerjee et al [61] gave two dynamical solutions

$$A_I = \frac{T(t) z^{1/2} - \alpha}{T(t) z^{1/2} + \alpha}, \quad A_{II} = \frac{z}{T(t) (z + a/T(t))}, \quad z = 1 + \eta(t) u. \quad (111)$$

In 1990 Knutsen [16] corrected and extended the Goldman potential method by studying the characteristics, the physical plausibility and the dynamical stability of the static models in terms of the Goldman-Knutsen generating function. He discussed the third Goldman solution and proposed a simpler one with  $g = au + b$ .

Another static solution was given by Burlankov [62]

$$L^2 = a \left( \frac{u + b - \sqrt{3}/2}{u + b + \sqrt{3}/2} \right)^{\sqrt{3}}. \quad (112)$$

Recently, Pant et al [63] studied in detail the static metric

$$A = \frac{\cos \sqrt{2}b(d - u)}{\sin(a + bu)}, \quad B = \sec^2(a + bu). \quad (113)$$

Finally, Msomi, Govinder and Maharaj [10], with the help of Lie symmetry analysis, found five transformations, leading to new solutions from old ones and introduced their generating function  $W$ .

## 9 Conclusions

We have given a global view upon the study of collapsing shear-free perfect fluid spheres with heat flow. The application of the LG formalism has been advocated throughout the present paper. It provides a very compact formulation of the isotropy condition (12), namely Eq (29), and a very simple expression for  $\Psi_2$  - Eq (56). The formulas for the other characteristics, Eqs (52-59), are also straight and tractable. Eq (56) clearly shows why the condition for conformal flatness is so similar to the isotropy condition.

The LG formalism also presents the simplest possible version of the junction condition. This has been demonstrated explicitly for conformally flat and for geodesic solutions. It gives the right functions to disentangle this condition into well known differential equations like the Abel equation, the Riccati equation, the Bernoulli equation or the linear one. This formalism yields an alternative derivation of the general solution when the metric components are functionally dependent.

We have also discussed an unified study of separable solutions by incorporating the simple linear in time ansatz into the general formula for the solution of the junction condition (76).

One of the main objectives of the paper is to bring together the results of the static and dynamical group of authors, not only in the chronology of particular solutions, but in the discovery of generating functions. The recent proposition of the generating potential  $W$  [10] has prompted the search for similar functions, mainly in the work of the static group. A bunch of five generating potentials has been found, any of which provides the complete solution of Eq (29). Their common feature is the presence of the basic form  $AB$ . Its use reduces the LG equation to a first order or an algebraic one, depending on which potential of the pair is chosen as a seed. It seems that the Goldman-Knutsen generating function satisfies the simplest equation and a future task may be to continue the studies of Knutsen upon characteristics, pertinent to the dynamical models. Obviously, new generating potentials may be proposed by taking other functions of  $AB$  or its  $u$ -derivative. In view of this we hope that the enumeration of the existing generating functions, undertaken here, will prevent wasting of time in rediscoveries in the future.

Finally, putting in order the four-dimensional case will help to lessen the efforts in investigating shear-free radiating collapse in higher dimensions [11],[36].

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